On the maximal multiplicity of long zero-sum free sequences over $C_p \oplus C_p$

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Abstract

In this paper, we point out that the method used in [Acta Arith. 128(2007) 245-279] can be modified slightly to obtain the following result. Let $\varepsilon \in (0, \frac{1}{4})$ and $c > 0$, and let p be a sufficiently large prime depending on ε and c. Then every zero-sumfree sequence S over $C_p \oplus C_p$ of length $|S| \geq 2p - c\sqrt{p}$ contains some element at least $\lfloor p^{\frac{1}{4}-\varepsilon} \rfloor$ times.

Keywords: zero-sumfree, multiplicity.

1. Introduction

The structure of long zero-sumfree sequences over a finite cyclic group has been well studied since 1975 (See [1], [7],[14], [15] and [11]). For example, it has been proved by Savchev and Chen [14], and by Yuan [15] independently, that every zero-sumfree sequence over C_n of length at least $\frac{n}{2} + 1$ is a partition (up to an integer factor co-prime to n) of a positive integer smaller than n. But for the group $G = C_n \oplus C_n$, the structure of zero-sumfree sequences S over G has been determined so far only for the case that S is of the maximal length $2n - 2$. In 1969, Emde Boas and Kruyswijk [3] conjectured that every minimal zero-sum sequence over $C_p \oplus C_p$ of length $2p - 1$ contains some element $p - 1$ times, and in 1999, Gao and Geroldinger [6] conjectured that the same result holds true for any group $C_n \oplus C_n$. It is easy to see that the above conjecture is equivalent to that every zero-sum free sequence S over G of length $2n - 2$ contains some element at least $n - 2$

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times, which implies a complete characterization of the structure of S. This conjecture has been solved quite recently by combing two results obtained by Reiher [13], and by Gao, Geroldinger and Grynkiewicz [10]. Reiher [13] used about forty pages to prove that the above conjecture is true for every prime p, and Gao, Geroldinger and Grynkiewicz [10] used more than fifty pages to prove that the above conjecture is multiple, i.e., if it is true for $n = k$ and $n = \ell$ then it is also true for $n = k\ell$. Unlike the case for cyclic groups, we even can't determine the structure of zero-sumfree sequences over $C_n \oplus C_n$ of length $2n - 3$. In this paper we shall prove the following results by modifying the method used in [9].

Theorem 1.1 *Let $\varepsilon \in (0, \frac{1}{4})$ and $c > 0$, and let p be a sufficiently large prime depending on ε and c. If S is a zero-sumfree sequence over $C_p \oplus C_p$ of length $|S| \geq 2p - c\sqrt{p}$, then S contains some element at least $\lfloor p^{\frac{1}{4}-\varepsilon} \rfloor$ times.*

Theorem 1.2 *Let $\varepsilon \in (0, \frac{1}{4})$ and $c > 0$, and let p be a sufficiently large prime depending on ε and c. Let S be a sequence over $C_p \oplus C_p$ of length $|S| \geq 3p - c\sqrt{p} - 1$. If S contains no short zero-sum subsequence then S contains some element at least $\lfloor p^{\frac{1}{4}-\varepsilon} \rfloor$ times.*

Theorem 1.3 *Let $\varepsilon \in (0, \frac{1}{4})$ and $c > 0$, and let p be a sufficiently large prime depending on ε and c. Let S be a sequence over $C_p \oplus C_p$ of length $|S| \geq p^2 + 2p - c\sqrt{p} - 1$. If S contains no zero-sum subsequence of length p^2 then S contains some element at least $\lfloor p^{\frac{1}{4}-\varepsilon} \rfloor$ times.*

2. Notations

Our notation and terminology are consistent with [9]. We briefly gather some key notions and fix the notations concerning sequences over finite abelian groups. Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{P}$ the set of prime integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any two integers $a, b \in \mathbb{N}$, we set $[a, b] = \{x \in \mathbb{N} : a \leq x \leq b\}$. Throughout this paper, all abelian groups will be written additively, and for $n, r \in \mathbb{N}$, we denote by C_n the cyclic group of order n, and denote by C_n^r the direct sum of r copies of C_n.

Let G be a finite abelian group and $\exp(G)$ its exponent. Let $\mathcal{F}(G)$ be the free abelian monoid, multiplicatively written, with basis G. The elements of $\mathcal{F}(G)$ are called sequences over G. We write sequences $S \in \mathcal{F}(G)$ in the form

$$S = \prod_{g \in G} g^{v_g(S)}, \text{ with } v_g(S) \in \mathbb{N}_0 \text{ for all } g \in G.$$

We call $v_g(G)$ the multiplicity of g in S, and we say that S contains g if $v_g(S) > 0$. Further, S is called squarefree if $v_g(S) \leq 1$ for all $g \in G$. The unit element $1 \in \mathcal{F}(G)$ is called the empty sequence. A sequence S_1 is called a subsequence of S if $S_1 \mid S$ in $\mathcal{F}(G)$. For a subset A of G we denote $S_A = \prod_{g \in A} g^{v_g(S)}$. If a sequence $S \in \mathcal{F}(G)$ is written in the form $S = g_1 \cdot \ldots \cdot g_l$, we tacitly assume that $l \in \mathbb{N}_0$ and $g_1, \ldots, g_l \in G$.

For a sequence

$$S = g_1 \cdot \ldots \cdot g_l = \prod_{g\in G} g^{v_g(S)} \in \mathcal{F}(G),$$

we call

- $|S| = l = \sum_{g\in G} v_g(G) \in \mathbb{N}_0$ the *length* of S,
- $h(S) = \max\{v_g(S)|g \in G\} \in [0, |S|]\}$ the *maximum* of the *multiplicities* of S,
- $\mathrm{supp}(S) = \{g \in G|v_g(S) > 0\} \subset G$ the *support* of S,
- $\sigma(S) = \sum_{i=1}^{l} g_i = \sum_{g\in G} v_g(S)g \in G$ the *sum* of S,
- $\sum_k(S) = \{\sum_{i\in I} g_i | I \subset [1, l] \text{ with } |I| = k\}$ the *set* of $k-term\ subsums$ of S, for all $k \in \mathbb{N}$,
- $\sum_{\le k}(S) = \bigcup_{j\in[1,k]} \sum_j(S),\ \sum_{\ge k}(S) = \bigcup_{j\ge k} \sum_j(S)$,
- $\sum(S) = \sum_{\ge 1}(S)$ the *set* of all *subsums* of S.

The sequence S is called

- a $zero-sum\ sequence$ if $\sigma(S) = 0$.
- $zero-sumfree$ if $0 \notin \sum(S)$.

3. Proof of the main results

Lemma 3.1 *[12, Lemma 2.6] Let G be prime cyclic of order $p \in \mathbb{P}$ and S a sequence in $\mathcal{F}(G)$. If $v_0(S) = 0$ and $|S| = p$, then $\sum_{\le h(S)}(S) = G$.*

Lemma 3.2 *[2] Let G be prime cyclic of order $p \in \mathbb{P}$, $S \in \mathcal{F}(G)$ a squarefree sequence and $k \in [1, |S|]$.*

(1) $|\sum_k(S)| \ge \min\{p, k(|S| - k) + 1\}$*;*

(2) If $k = \lfloor |S|/2 \rfloor$*, then* $|\sum_k(S)| \ge \min\{p, (|S|^2 + 3)/4\}$*;*

(3) If $|S| = \lfloor \sqrt{4p - 7} \rfloor + 1$ *and* $k = \lfloor |S|/2 \rfloor$*, then* $\sum_k(S) = G$*.*

Lemma 3.3 *[9, Lemma 4.2] Let $G = C_p \oplus C_p$ with $p \in \mathbb{P}$, (e_1, e_2) a basis of G and*

$$S = \Pi_{i=1}^{l}(a_i e_1 + b_i e_2) \in \mathcal{F}(G),\ \textit{where}\ a_1, b_1, \ldots, a_l, b_l \in \mathbb{F}_p,$$

a zero-sumfree sequence of length $|S| = l \ge p$. Then

$$\Big|\{\sum_{i\in I} b_i | \emptyset \ne I \subset [1, l]\ \textit{with}\ \sum_{i\in I} a_i = 0\}\Big| \ge l - p + 1.$$

Lemma 3.4 *Let $\varepsilon \in (0, \frac{1}{2})$, $c > 0$ and $1 < r \in \mathbb{N}$, and let p be a sufficiently large prime depending on ε, c and r. Let $G = C_p^r$, and let S be a sequence over G of length $|S| \geq p$. Suppose that $|S_{g+H}| \leq \lfloor cp^{\frac{1}{2}-\varepsilon} \rfloor$ holds for all subgroups H of order p^{r-1} and all $g \in G$. Then $0 \in \sum(S)$.*

Proof. Let p be a sufficiently large prime depending on ε, c and r. Assume to the contrary that there exists a zero-sumfree sequence

$$S = \Pi_{i=1}^{s} g_i \in \mathcal{F}(G) \text{ of length } |S| = s \geq p$$

and such that

$$\left|S_{g+H}\right| \leq \lfloor cp^{\frac{1}{2}-\varepsilon} \rfloor \text{ for any subgroup } H \text{ of order } p^{r-1} \text{ and any } g \in G.$$

Let $\hat{G} = \mathrm{Hom}(G, \mathbb{C}^{\times})$ be the character group of G with complex values, $\chi_0 \in \hat{G}$ the principal character, and for any $\chi \in \hat{G}$ let

$$f(\chi) = \prod_{i=1}^{s}(1 + \chi(g_i)).$$

Clearly, we have $f(\chi_0) = 2^s$ and

$$f(\chi) = 1 + \sum_{g \in \sum(S)} c_g \chi(g),$$

where $c_g = |\{\emptyset \neq I \subset [1, s] | \sum_{i \in I} g_i = g\}|$. Since S is zero-sumfree, we have $0 \notin \sum(S)$ and the Orthogonality Relations(see [[8], Lemma 5.5.2]) imply that

$$\sum_{\chi \in \hat{G}} f(\chi) = \sum_{\chi \in \hat{G}}(1 + \sum_{g \in \sum(S)} c_g \chi(g)) = |\hat{G}| + \sum_{g \in \sum(S)} c_g \sum_{\chi \in \hat{G}} \chi(g) = |G|.$$

Let $\chi \in \hat{G} \setminus \{\chi_0\}$. We set $M = \lfloor cp^{\frac{1}{2}-\varepsilon} \rfloor$ and

$$|S| = (2k-1)M + q \text{ with } q \in [0, 2M-1],$$

and continue with the following assertion:

A1. $|f(\chi)| \leq 2^s \exp(-\pi^2 v/(2p^2))$ with $v = 2M(1^2 + 2^2 + \cdots + (k-1)^2) + qk^2$.

Proof of A1. Let $j \in [-(p-1)/2, (p-1)/2]$ and $g \in G$ with $\chi(g) = \exp(2\pi i j/p)$. Note that for any real x with $|x| < \pi/2$, we have $\cos x \leq \exp(-x^2/2)$. Thus

$$|1 + \chi(g)| = 2\cos(\frac{\pi j}{p}) \leq 2\exp(\frac{-\pi^2 j^2}{2p^2}). \tag{1}$$

If $H = \mathrm{Ker}(\chi)$, then $|H| = p^{r-1}$ and $g + H = \chi^{-1}(\exp(2\pi i j/p))$. Thus applying

$$\left|S_{g+H}\right| \leq M$$

there are at most M elements $h \mid S$ such that $\chi(h) = \exp(2\pi i j/p)$. Consequently, the upper bound for $f(\chi)$, obtained by repeated application of (1), is maximal if the values $0, 1, -1, \ldots, k-1, -(k-1)$ are

accepted M times each and the values $k, -k$ are accepted q times as images of $\chi(g)$ for $g \in \mathrm{supp}(S)$. Therefore

$$|f(\chi)| \le 2^s \exp(-\pi^2 v/(2p^2)).$$

Since $|S| = s = (2k-1)M + q$, we get $k = \frac{s-q+M}{2M}$ and hence

$$\begin{aligned} v =& 2M \sum_{j=1}^{k-1} j^2 + qk^2 = 2M\frac{(k-1)(2k-1)k}{6} + qk^2 \\ =& \frac{(s-q-M)(s-q+M)(s-q) + 3q(s-q+M)^2}{12M^2}. \end{aligned}$$

Since $q \in [0, 2M-1]$ and $q \le s$, it follows that

$$v = \frac{s(s^2 - M^2)}{12M^2} + \frac{q(2M-q)(2M+3s-2q)}{12M^2} \ge \frac{s(s^2-M^2)}{12M^2}.$$

We deduce that (here we need p sufficiently large)

$$\exp(\frac{\pi^2 v}{2p^2}) \ge \exp(\frac{\pi^2 s(s^2-M^2)}{24M^2p^2}) > 2p^r, \tag{2}$$

where the last inequality holds because $s \ge p$ and p is sufficiently large and then $s^2 - M^2 > \frac{p^2}{2}$ and

$$\frac{\pi^2 s(s^2-M^2)}{24M^2p^2} > \frac{\pi^2 p^{2\varepsilon}}{2 \cdot 24c^2} > \ln(2p^r).$$

Therefore it follows that

$$\begin{aligned} p^r = |G| = \sum_{\chi \in \hat{G}} f(\chi) \ge& f(\chi_0) - \sum_{\chi \ne \chi_0} |f(\chi)| \\ \ge& 2^s(1 - (p^r - 1)\exp(\frac{-\pi^2 v}{2p^2})) > 2^s(1 - \frac{p^r-1}{2p^r}) > 2^{s-1} > p^r, \end{aligned}$$

a contradiction.

$\square$

Proof of Theorem 1.1. We may assume that $c > 8$. Let (e_1, e_2) be a basis of G and for $i \in [1, 2]$ let $\varphi_i : G \to \langle e_i \rangle$ denote the canonical projections. Let $\varepsilon > 0$, and let p be sufficiently large and assume to the contrary that there exists a zero-sumfree sequence

$$S = \Pi_{i=1}^{|S|}(a_i e_1 + b_i e_2) \in \mathcal{F}(G), \text{ with } a_1, b_1, \dots, a_s, b_s \in [0, p-1]$$

of length $|S| = s \ge 2p - c\sqrt{p}$ and with $h(S) \le p^{\frac{1}{4}-\varepsilon}$. Let T denote a maximal squarefree subsequence of S and set $h_0 = h(\varphi_1(T))$. After renumbering if necessary we may assume that

$$T = \Pi_{i=1}^{|T|}(a_i e_1 + b_i e_2), \ a_1 = \dots = a_{h_0} = a.$$

Now we set
$$W = \Pi_{i=1}^{h_0}(ae_1 + b_ie_2),\ S_1 = SW^{-1}$$
and distinguish three cases.

Case 1: $h_0 \geq \lfloor\sqrt{4p-7}\rfloor + 1$. We set $k = \lfloor\sqrt{4p-7}\rfloor + 1,\ l = \lfloor k/2 \rfloor$ and
$$S_2 = \Pi_{i=k+1}^{s}(a_ie_1 + b_ie_2).$$
By Lemma 3.2(3) we have
$$\textstyle\sum_l(\Pi_{i=1}^{k} b_ie_2) = \langle e_2\rangle. \tag{3}$$
Consider the sequence $\varphi_1(S_2) = \Pi_{i=k+1}^{s} a_ie_1$. Let $v_0(\varphi_1(S_2)) = t$ and after renumbering if necessary we may set
$$W_1 = \Pi_{i=k+1}^{k+1+t}(0e_1 + b_ie_2),\ W_1 \mid S_2.$$

Since W_1 is zero-sumfree, the sequence $\varphi_2(W_1) = \Pi_{i=k+1}^{k+1+t} b_ie_2$ is a zero-sumfree sequence over C_p. It follows from Lemma 3.2(3) that $|\operatorname{supp}(\varphi_2(W_1))| \leq \lfloor\sqrt{4p-7}\rfloor$. By the contrary hypothesis we have that $h(\varphi_2(W_1)) = h(W_1) \leq h(S) < p^{\frac{1}{4}}$. Therefore, $t = |\varphi_2(W_1)| \leq h(\varphi_2(W_1))|\operatorname{supp}(\varphi_2(W_1))| \leq p^{\frac{1}{4}}\lfloor\sqrt{4p-7}\rfloor$. Hence,
$$|\varphi_1(S_2)| - v_0(\varphi_1(S_2)) = s - k - t > 2p - c\sqrt{p} - (\lfloor\sqrt{4p-7}\rfloor + 1) - p^{\frac{1}{4}}\lfloor\sqrt{4p-7}\rfloor \geq p.$$
Thus Lemma 3.1 implies that $\sum(\varphi_1(S_2)) = \langle e_1\rangle$. In particular, S_2 has a non-empty subsequence S_3 such that $\sigma(\varphi_1(S_3)) = -lae_1$. By equation (3) there is a subset $I \subset [1,k]$ such that $\sum_{i\in I} b_ie_2 = -\sigma(\varphi_2(S_3))$ and $|I| = l$. Therefore, $S_3\Pi_{i\in I}(ae_1 + b_ie_2)$ is a non-empty zero-sum subsequence of S, a contradiction.

Case 2: $cp^{\frac{1}{4}} \leq h_0 \leq \lfloor\sqrt{4p-7}\rfloor$. Setting $k = \lfloor h_0/2\rfloor$ and $h_1 = h(\varphi_1(S_1))$ then Lemma 3.2(2) implies that
$$|\textstyle\sum_k(\Pi_{i=1}^{h_0} b_ie_2)| \geq \frac{h_0^2+3}{4} \tag{4}$$
and by the assumption of Case 2 we get
$$h_1 \leq h(\varphi_1(T))h(S) < h_0p^{1/4}.$$
Therefore,
$$|\varphi_1(S_1)| - v_0(\varphi_1(S_1)) \geq |S_1| - h_1 > 2p - cp^{1/2} - h_0 - h_0p^{1/4} \geq p - 1,$$
whence Lemma 3.1 implies $\sum_{\leq h_1}(\varphi_1(S_1)) = \langle e_1\rangle$. In particular, S_1 has a non-empty subsequence S_4 such that
$$\sigma(\varphi_1(S_4)) = -kae_1,\ |S_4| \leq h_1. \tag{5}$$
By equations (4) and (5) we infer that
$$\sigma(S_4) + \textstyle\sum_k(W) \subset \langle e_2\rangle,\ \left|\sigma(S_4) + \sum_k(W)\right| \geq \frac{h_0^2+3}{4}. \tag{6}$$
Set $S_5 = S(S_4W)^{-1}$. By Lemma 3.3 we have
$$\left|\sum(S_5) \cap \langle e_2\rangle\right| \geq |S_5| - p + 1.$$

Therefore, since $c > 8$ and p is sufficiently large,

$$\begin{aligned}
&\left|\sigma(S_4) + \textstyle\sum_k(W)\right| + \left|\textstyle\sum(S_5) \cap \langle e_2\rangle\right| \geq \tfrac{h_0^2+3}{4} + |S_5| - p + 1 \\
&\geq \frac{h_0^2+3}{4} + 2p - cp^{1/2} - h_0p^{1/4} - h_0 - p + 1 \\
&\geq \frac{h_0^2+3}{4} - h_0(p^{1/4}+1) - cp^{1/2} + 1 + p \\
&= h_0(\frac{1}{4}h_0 - p^{1/4} - 1) - cp^{1/2} + \frac{7}{4} + p \\
&\geq cp^{1/4}((\frac{c}{4} - 1)p^{1/4} - 1) - cp^{1/2} + \frac{7}{4} + p \geq p.
\end{aligned}$$

It follows from the Cauchy-Davenport theorem that

$$(\sigma(S_4) + \textstyle\sum_k(W)) + (\sum(S_5) \cap \langle e_2\rangle) = \langle e_2\rangle,$$

whence $0 \in \sigma(S_4) + \sum_k(W)) + (\sum(S_5) \cap \langle e_2\rangle) \subset \sum(S)$, a contradiction.

Case 3: $h_0 < cp^{1/4}$. Note that $|\operatorname{supp}(S) \cap (ae_1 + \langle e_2\rangle)| = h_0$. Thus we may suppose that, for every subgroup $H \subset G$ with $|H| = p$ and every $g \in G$, we have

$$|S_{g+H}| \leq h_0h(S) \leq \lfloor cp^{\frac{1}{2}-\varepsilon}\rfloor,$$

since otherwise we choose a different basis (e_1', e_2') of G and are back to Case 1 or Case 2. Therefore applying Lemma 3.4 with $r = 2$ we deduce that S is not zero-sumfree, a contradiction. □

Lemma 3.5 *([5], Theorem 6.7) Every sequence over $C_n \oplus C_n$ of length $3n - 2$ contains a zero-sum subsequence of length n or $2n$.*

Proof of Theorem 1.2. Let $k = 3p - 2 - |S|$. Then,

$$k \leq \lfloor c\sqrt{p}\rfloor - 1 < p.$$

Let $W = 0^kS$. Then, W is a sequence over $C_p \oplus C_p$ of length $|W| = 3p - 2$. By Lemma 3.6, W contains a zero-sum sequence T of length p or $2p$. So, $T_1 = T0^{-v_g(T)}$ is a nonempty zero-sum subsequence of S. Since S contains no short zero-sum subsequence, we infer that $|T_1| > p$ and $|T| = 2p$, and T_1 is minimal zero-sum. It follows that $2p \geq |T_1| \geq 2p - k \geq 2p - \lfloor c\sqrt{p}\rfloor + 1$. Take an arbitrary element $g|T_1$. Therefore, T_1g^{-1} is zero-sum free and $h(T_1g^{-1}) \geq p^{\frac{1}{4}-\varepsilon}$ by Theorem 1.1. □

Lemma 3.6 *([4], Theorem 2) Let G be a finite abelian group, and let S be a sequence over G of length $|S| = |G| + k$ with $k \geq 1$. If S contains no zero-sum subsequence of length $|G|$, then there exist a subsequence $T|S$ of length $|T| = k + 1$ and an element $g \in G$ such that $g + T$ is zero-sum free.*

Proof of Theorem 1.3. By Lemma 3.6, there exist a subsequence $T|S$ and an element $g \in C_p \oplus C_p$ such that $g + T$ is zero-sum free and $|g + T| = |T| = |S| - p^2 + 1 \geq 2p - c\sqrt{p}$. It follows from Theorem 1.1 that $h(S) \geq h(T) = h(g + T) \geq p^{\frac{1}{4}-\varepsilon}$. □

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